Solution 1 by Michel Bataille, Rouen, France. The inequality is obvious if $ab(a-b)+bc(b-c)+ca(c-a)\le 0$ and otherwise is equivalent to

$$54\left((a-b)(b-c)(a-c)\right)^2 \le \left((a-b)^2+(b-c)^2+(c-a)^2\right)^3 \qquad (1)$$

(since $ab(a-b)+bc(b-c)+ca(c-a)=(a-b)(b-c)(a-c)$). Let $\mathcal{L}(a,b,c)=54\left((a-b)(b-c)(a-c)\right)^2$ and $\mathcal{R}(a,b,c)=\left((a-b)^2+(b-c)^2+(c-a)^2\right)^3$. If $a_1=a-c,\ b_1=b-c$ and $c_1=0$, then $a_1-b_1=a-b,\ b_1-c_1=b-c,\ a_1-c_1=a-c$ so that $\mathcal{L}(a_1,b_1,c_1)=\mathcal{L}(a,b,c)$ and $\mathcal{R}(a_1,b_1,c_1)=\mathcal{R}(a,b,c)$. It follows that it suffices to prove (1) in the case when $c=0$, that is, to show that $54(a-b)^2a^2b^2\le\left((a-b)^2+b^2+a^2\right)^3$ or equivalently,

$$27a^2b^2(a-b)^2\le 4(a^2+b^2-ab)^3. \qquad (2)$$

Now, it is straightforward to check the identity

$$4(a^2+b^2-ab)^3-27a^2b^2(a-b)^2=(a-2b)^2(2a-b)^2(a+b)^2$$

so that (2) writes as $(a-2b)^2(2a-b)^2(a+b)^2\ge 0$ and clearly holds.

Solution 2 by Arkady Alt, San Jose, California, USA. Due to cyclic symmetry of inequality we may assume that $a=\max\{a,b,c\}$. Since the inequality is obviously holds if $b<c$ (because then
$ab(a-b)+bc(b-c)+ca(c-a)=(a-b)(a-c)(b-c)\le 0$) suffice to consider only case when $b\ge c$, that is $a\ge b\ge c$. Let $x=b-c, y=a-b, p=x+y, q=xy$. Then $x,y\ge 0, a=c+x+y, b=c+x$,
$ab(a-b)+bc(b-c)+ca(c-a)=(x+y)\,xy=pq, (a-b)^2+(b-c)^2+(c-a)^2=\left(x^2+y^2+(x+y)^2\right)=2\left(x^2+y^2+xy\right)=2\left(p^2-q\right)$ and in the new notation the inequality is

$9\sqrt{2}pq\le\sqrt{3}\left(2\left(p^2-q\right)\right)^{3/2}$, where $q\ge 0$ and $q\le\dfrac{p^2}{4}$ (condition of solvability of Vieta's System $\begin{cases} x+y=p\\ xy=q\end{cases}$ in real x,y). We have $\sqrt{3}\left(2\left(p^2-q\right)\right)^{3/2}-9\sqrt{2}pq\ge$

$$\sqrt{3}\left(2\left(p^2-\frac{p^2}{4}\right)\right)^{3/2}-9\sqrt{2}p\cdot\frac{p^2}{4}=\sqrt{3}\left(\frac{3p^2}{2}\right)^{3/2}-\frac{9p^3}{2\sqrt{2}}=\frac{9p^3}{2\sqrt{2}}-\frac{9p^3}{2\sqrt{2}}=0.$$

Also solved by Kevin Soto Palacios, Huarmey,Peru; Ravi Prakash, New Delhi, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania and the proposer.

64. *Problem proposed by Arkady Alt, San Jose, California, USA.* Let $\Delta(x,y,z):=2(xy+yz+xz)-(x^2+y^2+z^2)$ and let a,b,c be sidelengths of a triangle with area F. Prove that

$$\Delta\left(a^3,b^3,c^3\right)\le\frac{64F^3}{\sqrt{3}}.$$

Solution by Michel Bataille, Rouen, France. In the featured solution of problem 1973 in *Mathematics Magazine*, Vol. 89, No 4, October 2016, p. 297, it is proved that

$$\Delta(a,b,c)\cdot\Delta(a^3,b^3,c^3)\le\left(\Delta(a^2,b^2,c^2)\right)^2 \qquad (1)$$

whenever a,b,c are positive real numbers. Taking for a,b,c the sidelengths of the triangle, we calculate

$$\Delta(a,b,c)=2(ab+bc+ca)-(a^2+b^2+c^2)=2(s^2+r^2+4rR)-(2s^2-2r^2-8rR)=4r(r+4R)>0$$

where s, r, R are the semi-perimeter, the inradius, the circumradius of the triangle, respectively, and

$$\Delta(a^2,b^2,c^2) = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4) = 16F^2$$

(from Heron's formula). Applying (1), we deduce

$$\Delta(a^3,b^3,c^3) \le \frac{64F^4}{r(r+4R)}$$

and see that it is sufficient to show that $\sqrt{3}F \le r(r+4R)$ or, since $F = rs$,

$$\sqrt{3}s \le r + 4R.$$

We are done since the latter is a known inequality, proved in O. Bottema *et al.*, *Geometric Inequalities*, Wolters-Noordhoff Publishing, 1968, **5.5**, p. 49.

Also solved by the proposer.

65. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Find all function $f : \mathbb{N} \to \mathbb{N}$ such that $mf(n) + f(m)$ is divisible by $f(m)(f(n)+1)$ for all $m, n \in \mathbb{N}$.

Solution by Michel Bataille, Rouen, France. The identity function $\mathrm{id}_{\mathbb{N}}$, defined by $\mathrm{id}_{\mathbb{N}}(n) = n$ for all $n \in \mathbb{N}$, is clearly a solution. We show that there are no other solutions. To this end, we consider an arbitrary solution f and prove that we must have $f(m) = m$ for all $m \in \mathbb{N}$. For each pair $(m, n) \in \mathbb{N} \times \mathbb{N}$, we have

$$mf(n) + f(m) = g(m,n)f(m)(f(n)+1) \tag{1}$$

for some positive integer $g(m, n)$.
Let $a = f(1)$. With $(m, n) = (1, 1)$, (1) yields $2a = g(1,1)a(a+1)$, hence $2 = (a+1)g(1,1)$ and so $a+1 = 2$, that is, $f(1) = 1$. From (1), we then deduce that

$$(2g(m,1)-1)f(m) = m \tag{2}$$

for any positive integer m. Consider any $m > 1$; such an integer can be written as $m = 2^r \cdot s$ for a unique pair (r, s) where r is a nonnegative integer and s is a positive odd integer. Using (2), we obtain $(2g(m,1)-1)f(2^r s) = 2^r s$ or, setting $f(2^r s) = 2^{r'}s'$ ($r' \ge 0, s'$ odd), $(2g(m,1)-1)2^{r'}s' = 2^r s$. This demands $r' = r$ and $s' = d$, some divisor of s, so that $f(m) = f(2^r s) = 2^r d$ where $s = dd'$ for integers d, d'. Note that in particular $f(2^r) = 2^r$.
Now, equality (1) with $m = 2^r s$ and $n = 2^u$ ($u \in \mathbb{N}$) gives $2^u d' + 1 = g(m,n)(2^u+1)$. As a result, the integer 2^u+1 divides $2^u d' + 1 = (2^u+1)d' + 1 - d'$, hence also divides $d' - 1$. Since u is arbitrary, $d' - 1$ has infinitely many divisors. The only possibility is $d' = 1$ and so $f(2^r s) = 2^r s$. The desired result $f(m) = m$ follows and the proof is complete.

Also solved by the proposer.

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